Topological quantization of the harmonic oscillator

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Abstract. We present a derivation of the energy spectrum of the harmonic oscillator by using the alternative approach of topological quantization. The spectrum is derived from the topological invariants of a particular principal fiber bundle which can be assigned to any configuration of classical mechanics, when formulated according to Maupertuis formalism.

1 Introduction

Although the idea of using topology to find quantum properties of classical systems is not new and has been used very actively in the context of diverse monopole and instanton configurations [1], a strict proof of the existence of the underlying fiber bundle structure was provided only recently for classical gravitational fields [2]. In an attempt to apply topological quantization to classical systems with only a finite number of degrees of freedom, we found that it is necessary to reformulate classical mechanics in such a way that the theorem proved in [2] can be applied. Fortunately, that formulation already exists and is known as Maupertuis approach. To show the applicability of the main theorem of topological quantization in this case, one needs to solve several technical problems related to the structure of principal fiber bundles with specific non compact fibers. This task is still under

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investigation. In this paper we will present a simple example in which the idea of topological quantization is applied to a simple harmonic oscillator. We will show that in fact it is possible to derive the energy spectrum of the harmonic oscillator by analyzing the topological invariants of the corresponding principal fiber bundle.

2 The approach of topological quantization

The method of topological quantization can be applied to any field configuration whose geometrical structure allows the existence of a principal fiber bundle. In the case of gravitational systems with an infinite number of degrees of freedom, a theorem proves the existence and uniqueness of such a bundle [2]. Indeed, the theorem states that any solution of Einstein's equations minimally coupled to any gauge matter field can be represented geometrically as a principal fiber bundle with spacetime as the base space. The structure group (isomorphic to the standard fiber) follows from the invariance of the metric of the base space with respect to Lorentz transformations, in the case of a vacuum solution, or with respect to a transformation of the gauge group, in the case of a gauge matter field. The topological invariants of the corresponding principal fiber bundle lead to a discretization of the parameters entering the metric of the base space. In the approach of topological quantization this is the kind of discretization we are interested in. For the proof of the above theorem it is very important that the base space is equipped with a metric whose invariance determines the standard fiber. In the search for a similar structure in classical mechanics we found that Maupertuis formalism provides a natural metric which can be used to fix the base space.

Consider a classical conservative system with n degrees of freedom described by the Lagrangian (summation over repeated indices)

$$L = \frac{1}{2} h_{ij} \dot{q}^i \dot{q}^j - V(q) \ . \tag{1}$$

Although the evolution of this system can be completely described within the Lagrangian formalism by varying the action $S = \int L dt$, we

will use Maupertuis formalism which is based upon the reduced action

$$S_0 = \int ds \;, \quad \text{with} \quad ds^2 = 2(E-V)h_{ij}dq^i dq^j \;, \tag{2}$$

where E is the total energy. The equations following from the variation $\delta S_0 = 0$, together with the expression for the time parameter in terms of the reduced action, completely describe the evolution of the system [3]. Equation (2) defines the natural metric $g_{ij} = 2(E-V)h_{ij}$ which we use to specify the base space B. In general, we can see that to any physical system in classical mechanics with n degrees of freedom corresponds an $n-$dimensional Riemannian space B with metric g_{ij}. The potential $V(q)$ is used to characterize different physical systems. Most systems in classical mechanics are invariant with respect to Galilean transformations. For the sake of simplicity, we limit ourselves here to systems which are invariant with respect to rotations only. Then to each point of the base space B we can associate a standard fiber $SO(n)$. Furthermore, if we identify the structure group G with the Lie group $SO(n)$, we have all the constituents of a $2n$-dimensional principal fiber bundle P. According to [2], the topological quantization of a classical physical system follows from the investigation of the topological invariants of P. In the present case, the only invariant characteristic class [4] is the Euler class $e(P)$ which is given in terms of the components of the curvature 2-form $R^i{}_j$ of B and whose integration yields an integer, say, n $(2m = n)$

$$\int e(P) = \frac{(-1)^m}{2^{2m}\pi^m m!}\int \varepsilon_{i_1 i_2 \ldots i_{2m}} \mathbf{R}^{i_1}{}_{i_2} \wedge \mathbf{R}^{i_3}{}_{i_4} \wedge \cdots \wedge \mathbf{R}^{i_{2m-1}}{}_{i_{2m}} = n \;. \tag{3}$$

Consider the simple example of a free particle, i.e. $V(q) = 0$. Then, the metric on the base space B is $g_{ij} = Eh_{ij}$ and the corresponding trajectories must be straight lines, i.e. the metric g_{ij} must be flat. This implies that there exists a coordinate system in which $h_{ij} = \delta_{ij}$ is the Euclidean metric. For zero curvature the Euler class vanishes and $n = 0$. We interpret this result as showing that a free particle is not quantized from the point of view of topological quantization. This is in accordance with the results of canonical quantization in quantum mechanics. In the general case $V(q) \neq 0$ we see that g_{ij} is a conformally flat metric, with conformal factor $2(E-V)$, for which clearly

the curvature is non zero, the Euler class does not vanish and, according to Eq.(3), the quantization is not trivial. In the following section we present an explicit example of non trivial quantization.

3 The harmonic oscillator

Consider two harmonic oscillators of the same mass m so that $h_{ij} = m\delta_{ij}$ and the metric components of the base space B read

$$g_{ij} = 2m\left[E - \frac{1}{2}k_1(q^1)^2 - \frac{1}{2}k_2(q^2)^2\right]\delta_{ij} := e^{\phi}\,\delta_{ij}\ . \qquad (4)$$

This physical system is invariant with respect to transformations of the group $SO(2)$ which is taken as the structure group and standard fiber of the principal bundle P. The corresponding Euler class can be expressed as (a coma denotes partial derivative)

$$e(P) = -\frac{1}{2\pi}R^1{}_2 = \frac{1}{4\pi}(\phi_{,11} + \phi_{,22})dq^1 \wedge dq^2\ . \qquad (5)$$

The calculation is straightforward, but the resulting expression is quite cumbersome. To simplify the analysis we consider the special case $k_2 = 0$, and let $k_1 = k$ and $q^1 = q$. Then

$$\int e(P) = -\frac{kb}{4\pi}\int \frac{E + \frac{1}{2}kq^2}{(E - \frac{1}{2}kq^2)^2}dq = n\ , \qquad (6)$$

where we choose as $b\pi$ the constant resulting from the integration over q^2. Integrating over q within the interval $[-q_0, q_0]$, we obtain

$$\frac{bq_0}{q_0^2 - a^2} = n\ , \qquad a^2 = \frac{2E}{k}\ , \qquad (7)$$

where a represents the classical turning point. This represents a relationship among the parameters describing the harmonic oscillator, i.e., the energy E, the constant k and q_0 that depends on the former two. This relationship is unique and gives us information about the discrete nature of the system from the point of view of topological quantization. We call this the topological spectrum of the harmonic oscillator.

On the other hand, the canonical formalism provides us with a unique canonical spectrum for the energy of the system, and we aim for a direct relation between this and the topological spectrum. This, however, requires an exact definition of quantum states in the context of topological quantization [5], which is beyond the scope of the present work. Nevertheless, a simple way to show the equivalence is to choose the limit of integration q_0 as

$$q_0 = \frac{1}{C} - \sqrt{\frac{1}{C^2} + a^2}\ , \quad C = \frac{2}{b}\left(\frac{E}{\hbar\omega} - \frac{1}{2}\right)\ , \quad \omega = \sqrt{\frac{k}{m}} \tag{8}$$

which for any positive finite value of C reduces the topological spectrum (7) to the canonical spectrum $E = \hbar\omega(n+1/2)$. An analysis of the limiting cases shows that the choice (8) is physically meaningful. Indeed, when $E >> \hbar\omega$ then $\frac{1}{C} \to 0$, and $q_0 \to a$, that is to say q_0 tends to the turning point and we recover the classical limit. Moreover, in the limit $E \to 0$, the turning point goes to zero and we have $q_0 \to 0$, as expected.

The above results show that it is possible to obtain the canonical energy spectrum of the harmonic oscillator by using the approach of topological quantization.

Acknowledgments

This work was supported by Conacyt, Mexico, grant 48601.

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